1 Optimization Problem

Suppose $Y \sim (\mu, 1)$ and let

$$L_q(\mu) = 2^{q-2}(Y-\mu)^2 + \lambda|\mu|^q$$

and

$$\hat{\mu}_q = \arg\min_{\boldsymbol{\mu}} L_q(\mu)$$

One approach to solve this optimization problem is $subdifferentiation$.

2 Subdifferential

Definition 2.1. *c is a* ***subderivative*** *of f at X_0 when:*

$$f(X) - F(X_0) \geq c(X - X_0)$$

Denote the set of subderivatves by $\partial f|_{X_0}$.

A convex function can be optimized by setting it's subderivative to zero.

3 ℓ_1 and Soft-Thresholding

$\hat{\mu}_1$ minimizes L_1 if and only if $0 \in L_1|_{\hat{\mu}_1}$

We define *soft thresholding* as follows:

$$\hat{\mu}_1 = \begin{cases} Y+\lambda; & Y < -\lambda \\ 0; & -\lambda \leq Y \leq \lambda = \text{sgn}(Y)(|Y|-\lambda)_+ \\ Y-\lambda; & Y > \lambda \end{cases}$$

4 Example of Orthogonal Design

Let $p \leq n$ and $\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\frac{1}{n}\mathbb{X}^T\mathbb{X} = \mathbf{I}$.

We solve the following minimization problem:

$$\hat{\boldsymbol{\beta}}_\lambda = \arg\min_{\boldsymbol{\beta}} \frac{1}{2n} \|\mathbb{X}\beta - \mathbf{Y}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1$$

This can be minimized component wise by minimizing $L(\beta) = \beta^2 - \beta\hat{\beta}_{LS} + \lambda|\beta|$

This can be optimized by subdifferentials.

5 Normal Means Problem

Let $\boldsymbol{\epsilon} \sim N(0,1)$, then

$$\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \Leftrightarrow W \overset{D}{=} \boldsymbol{\beta} + \frac{1}{\sqrt{n}}\boldsymbol{\epsilon}$$

Let

- $\mathcal{H}$ be a real, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
- $(\boldsymbol{\phi_i})$ be an orthonormal basis for $\mathcal{H}$.

Define *Gaussian process*:

$$Y(t)dt = h(t)dt + d\epsilon(t)$$

$$y_i = \langle \boldsymbol{Y}, \boldsymbol{\phi_i} \rangle = \langle \boldsymbol{h} + \boldsymbol{\epsilon}, \boldsymbol{\phi_i} \rangle = h_i + \epsilon_i$$